Topic → Linear Differential Equations with constant co-efficients.
(Method of finding particular integral)

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Subject - Maths (Hons)

Study Material

General Solution of

$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n) y = x$

General Solution = C.F + P.I

C.F = Complementary Function
P.I = Particular integral

Meaning of $\frac{1}{f(D)}$

$\frac{1}{f(D)} x$ is that function of x
Which when operated by $f(D)$ gives x

Thus $f(D) \frac{1}{f(D)} x = x$

Therefore, $f(D)$ and $\frac{1}{f(D)}$ are inverse operators.

Thus $\frac{1}{f(D)}$ stands for integration

$\frac{1}{f(D)} x$ is particular integral of $f(D) y = x$

1 working rule for finding particular integral

Let $f(D) = (D-\alpha_1)(D-\alpha_2) \cdots (D-\alpha_n)$

$$\frac{1}{f(D)} = \frac{A_1}{D-\alpha_1} + \frac{A_2}{D-\alpha_2} + \cdots + \frac{A_n}{D-\alpha_n}$$

Now particular integral

$$\frac{1}{f(D)} X = \left[\frac{A_1}{D-\alpha_1} + \frac{A_2}{D-\alpha_2} + \cdots + \frac{A_n}{D-\alpha_n}\right] X$$

$$= A_1 \frac{1}{D-\alpha_1} X + A_2 \frac{1}{D-\alpha_2} X + \cdots + A_n \frac{1}{D-\alpha_n} X$$

$$= A_1 e^{\alpha_1 x} \int e^{-\alpha_1 x} X \, dx + A_2 e^{\alpha_2 x} \int e^{-\alpha_2 x} X \, dx + \cdots$$

$$+ A_n e^{\alpha_n x} \int e^{-\alpha_n x} X \, dx$$

which can be evaluated and thus particular integral can be found.

$f(D) e^{ax} = f(a) e^{ax}$ provided $f(a) \neq 0$.

$\frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax$ Except $f(-a^2) \neq 0$

$\frac{1}{f(D^2)} \cos ax = \frac{1}{f(-a^2)} \cos ax$

Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \sin 2x$

Solution —

Given Equation in Symbolic form

$$(D^2 + D + 1)y = \sin 2x$$

Auxiliary Equation

$$D^2 + D + 1 = 0$$

$$\Rightarrow D = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \ -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$\therefore CF = e^{-\frac{x}{2}}\left[A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right)$$

$$PI = \frac{1}{D^2+D+1}\sin 2x$$

$f(D) = D^2 + D + 1$

$f(D^2) = -4 + D + 1$

$$= \frac{\sin 2x}{-4+D+1}$$

$$= \frac{\sin 2x}{D-3}$$

$$= \frac{(D+3)\sin 2x}{D^2-9}$$

$$= \frac{(D+3)\sin 2x}{-4-9}$$

$$= -\frac{1}{13}(D+3)\sin 2x$$

$$= -\frac{1}{13}\left[\frac{d}{dx}\sin 2x + 3\sin 2x\right]$$

$$= -\frac{1}{13}\left[2\cos 2x + 3\sin 2x\right]$$

Hence General solution $= CF + PI = e^{-x/2}\left[A\cos\frac{\sqrt{3}}{2}\right.$